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Bi-Graded Markovian Matrices as Non-Local Dirac Operators and a New Quantum Evolution.

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Abstract

Measuring distances on a lattice in noncommutative geometry involves square, symmetric and real "three-diagonal" matrices, with the sum of their elements obeying a supremum condition, together with a constraint forcing the absolute value of the maximal eigenvalue to be equal to 1. In even dimensions, these matrices are unipotent of order two, while in odd dimensions only their squares are Markovian. We suggest that these bi-graded Markovian matrices (i.e. consisting in the square roots of Markovian matrices) can be thought of as non-local Dirac operators. The eigenvectors of these matrices are spinors. Treating these matrices as determining the stochastic time evolution of states might explain why one observes only left handed neutrinos. Some other physical interpretations are suggested. We end by presenting a mathematical conjecture applying to $q-$ graded Markovian matrices.

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1 Bi-graded Markovian matrices as non-local Dirac operators

Measuring distances in noncommutative geometry involves the spectral triple $(A,\ H,\ D)$. Where $A-$ is the algebra of functions defined over the space, $H-$ is a Hilbert space of spinors, and $D-$ is a Dirac operator. The Dirac operator acts essentially on both the functional space and the Hilbert space. The distance in noncommutative geometry is defined by the following formula:

$$d(a,b) = \sup_f \{|f(a) - f(b)| \ :\ f \in A,\ \|\ [\ D, f\ ]\ \| \leq 1\} \tag{1}$$

This formula was applied to an infinite one-dimensional lattice. The lattice is defined as: $\{x_k = ka\ ,\ \ k \in \mathbb{Z}\}$, where a is the lattice constant (i.e. the quantity which carries the relevant physical unit). Finding the distances on a one-dimensional lattice was the goal of [1, 2]. In these works [1, 2], one uses the local discrete Dirac-Wilson operator, usually applied in lattice gauge theories[1]. The $\|\ [\ D, f\ ]\ \| \leq 1$ condition and the $\sup_f |f(a) - f(b)|$ condition of (eq.1) are here reformulated for the evaluation of real, square and symmetric "three-diagonal" matrices M_k of the following form:

$$M_k = \begin{pmatrix} 0 & \Delta_1 & 0 & 0 & 0 \\ \Delta_1 & 0 & \Delta_2 & 0 & 0 \\ 0 & \Delta_2 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \Delta_{k-1} \\ 0 & 0 & 0 & \Delta_{k-1} & 0 \end{pmatrix} \tag{2}$$

under the restrictions that the $\sum_i \triangle_i$ should be a supremum and that the maximal eigenvalue of $M_k - \ \lambda_{max} \leq 1$, where $\Delta_i = f_i - f_{i-1}$, with the norm of a matrix defined as the largest eigenvalue. The distance from some $0-$ site to the $(k-1)-$ site of the lattice is $2a \sum_i \triangle_i$. In this work, we assign to the matrix M_k as a whole an interpretation as a non-local Dirac operator. This interpretation will be further justified in the sequel, but we note, meanwhile, that M_k is constructed solely out of local Dirac-Wilson operators, acting both on the functional space and on the Hilbert space defined over the lattice. As the matrix M_k acts simultaneously on different sites (essentially k sites), it should be considered as a non-local operator.

The problem of finding the distances on a lattice was formulated in [1] and resolved in [1, 2]. What was proved in [2] is that the matrices are given (under the above restrictions) by the following formulae:

$$k = 2n : \quad \triangle_{2i-1}^{(k)} = 1\ \ ,\ \ \triangle_{2i}^{(k)} = 0$$

[1]In [1, 2] the Dirac-Wilson operator used was the discrete differentiation, defined as $\frac{f_{i+1} - f_{i-1}}{2a}$

$k = 2n + 1$:

$$\triangle_i^{(k)} = \frac{\frac{1}{2}\left(1-(-1)^i\right)(k+1) + 2\,(-1)^i \left[\frac{i+1}{2}\right]}{\sqrt{(k-1)\,(k+1)}} \quad \forall\, i \leq k-1 \tag{3}$$

where the [ ] brackets stand for the integer value of the term within.
The numbers for the first few odd cases are of the form:

$$k = 3 \ : \quad \left\{\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right\}$$

$$k = 5 \ : \quad \left\{\tfrac{2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}\right\}$$

$$k = 7 \ : \quad \left\{\tfrac{3}{\sqrt{12}}, \tfrac{1}{\sqrt{12}}, \tfrac{2}{\sqrt{12}}, \tfrac{2}{\sqrt{12}}, \tfrac{1}{\sqrt{12}}, \tfrac{3}{\sqrt{12}}\right\}$$

$$k = 9 \ : \quad \left\{\tfrac{4}{\sqrt{20}}, \tfrac{1}{\sqrt{20}}, \tfrac{3}{\sqrt{20}}, \tfrac{2}{\sqrt{20}}, \tfrac{2}{\sqrt{20}}, \tfrac{3}{\sqrt{20}}, \tfrac{1}{\sqrt{20}}, \tfrac{4}{\sqrt{20}}\right\}$$

As one can see, those sequences alternate in a very special way. This is illustrated for $k = 41$ in the following figure, (with a slight smoothening out):

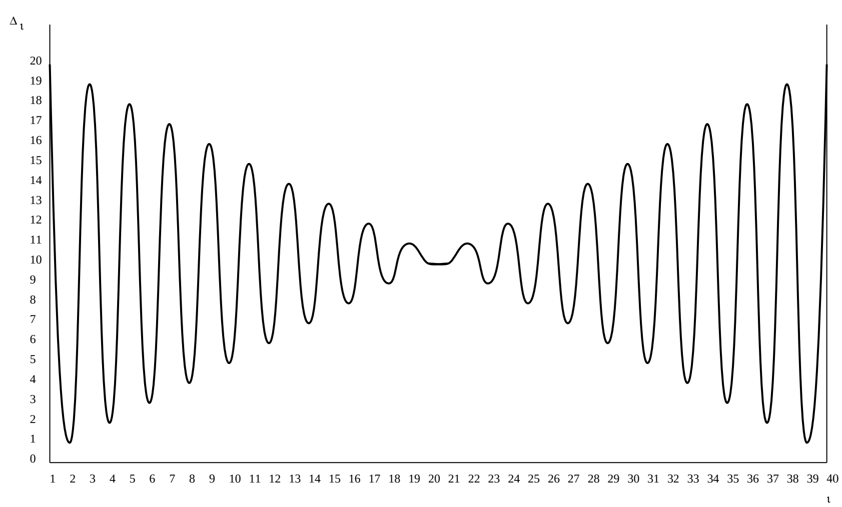

Figure 1.The values of Δ_i for $k = 41$ in units of $\frac{1}{\sqrt{420}}$.

These matrices have maximal eigenvalue 1, and the $\sum_i \triangle_i$ is a supremum. For k even, the matrices are unipotent of order two (therefore also Markovian). For k odd, however, only their squares are Markovian. We demonstrate these features for the the first few matrices:
$k = 2$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$k = 3:$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}^{2n} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}^{2n+1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$k = 4:$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{2n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{2n+1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$k = 5:$

$$\begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 0 \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} \\ 0 & 0 & 0 & \frac{2}{\sqrt{6}} & 0 \end{pmatrix}^{2} = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{5}{6} & 0 & \frac{1}{6} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{6} & 0 & \frac{5}{6} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 0 \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} \\ 0 & 0 & 0 & \frac{2}{\sqrt{6}} & 0 \end{pmatrix}^{2n\to\infty} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 0 \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} \\ 0 & 0 & 0 & \frac{2}{\sqrt{6}} & 0 \end{pmatrix}^{2n+1\to\infty} = \begin{pmatrix} 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 \end{pmatrix}$$

In the general case it will be:
$k = even$:

$$(M_k)^{2n} = \mathbb{I}_k$$

$$(M_k)^{2n+1} = M_k$$

$k = odd -$ These will have an asymptotic behavior:

$$(M_k)^{2n\to\infty} = \begin{pmatrix} \frac{2}{k+1} & 0 & \frac{2}{k+1} & 0 & \frac{2}{k+1} & \dots & \frac{2}{k+1} \\ 0 & \frac{2}{k-1} & 0 & \frac{2}{k-1} & 0 & \dots & 0 \\ \frac{2}{k+1} & 0 & \frac{2}{k+1} & 0 & \frac{2}{k+1} & \dots & \frac{2}{k+1} \\ 0 & \frac{2}{k-1} & 0 & \frac{2}{k-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 0 & \frac{2}{k-1} & 0 & \frac{2}{k-1} & 0 & \dots & 0 \\ \frac{2}{k+1} & 0 & \frac{2}{k+1} & 0 & \frac{2}{k+1} & \dots & \frac{2}{k+1} \end{pmatrix}$$

$$(M_k)^{2n+1\to\infty} = \begin{pmatrix} 0 & \frac{2}{\sqrt{(k^2-1)}} & 0 & \frac{2}{\sqrt{(k^2-1)}} & \dots & 0 \\ \frac{2}{\sqrt{(k^2-1)}} & 0 & \frac{2}{\sqrt{(k^2-1)}} & 0 & \dots & \frac{2}{\sqrt{(k^2-1)}} \\ 0 & \frac{2}{\sqrt{(k^2-1)}} & 0 & \frac{2}{\sqrt{(k^2-1)}} & \dots & 0 \\ \frac{2}{\sqrt{(k^2-1)}} & 0 & \frac{2}{\sqrt{(k^2-1)}} & 0 & \dots & \frac{2}{\sqrt{(k^2-1)}} \\ \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \frac{2}{\sqrt{(k^2-1)}} & 0 & \frac{2}{\sqrt{(k^2-1)}} & 0 & \dots & \frac{2}{\sqrt{(k^2-1)}} \\ 0 & \frac{2}{\sqrt{(k^2-1)}} & 0 & \frac{2}{\sqrt{(k^2-1)}} & \dots & 0 \end{pmatrix}$$

The eigenvalues of a bi-graded Markovian matrix are bounded by $-1 \leq \lambda_i \leq 1$, where the ± 1 eigenvalues always exist. In the $(k = 2n)-$ case, the ± 1 are the only eigenvalues

which exist (each of them with $n-$ degeneracy). In the odd case, the ± 1 eigenvalues appear with no degeneracy.
The corresponding eigenvectors of the ± 1 eigenvalues have the following normalized form in the $(k = 2n)-$ case:[2]

$$\begin{array}{c} \Big\{\pm 1 \leftrightarrow \Big(\pm\frac{1}{\sqrt{2}}\ ,\ \frac{1}{\sqrt{2}}\ ,\ 0\ ,\ \ldots\ ,\ 0\Big)\Big\} \\ \vdots \\ \Big\{\pm 1 \leftrightarrow \Big(0\ ,\ \ldots\ ,\ 0\ ,\ \pm\frac{1}{\sqrt{2}}\ ,\ \frac{1}{\sqrt{2}}\ ,\ 0\ ,\ \ldots\ ,\ 0\Big)\Big\} \\ \vdots \\ \Big\{\pm 1 \leftrightarrow \Big(0\ ,\ \ldots\ ,\ 0\ ,\ \pm\frac{1}{\sqrt{2}}\ ,\ \frac{1}{\sqrt{2}}\Big)\Big\} \end{array} \tag{4}$$

The corresponding eigenvectors of the ± 1 eigenvalues have the following normalized form in the $(k = 2n+1)-$ case:[3]

$$\begin{array}{l} \Big\{1 \leftrightarrow \Big(\frac{1}{\sqrt{k+1}}\ ,\ \frac{1}{\sqrt{k-1}}\ ,\ \cdots\ \frac{1}{\sqrt{k-1}}\ ,\ \frac{1}{\sqrt{k+1}}\Big)\Big\} \\ \Big\{-1 \leftrightarrow \Big(\frac{(-1)^n}{\sqrt{k+1}}\ ,\ \frac{(-1)^{n+1}}{\sqrt{k-1}}\ ,\ \ldots\ ,\ \frac{(-1)^{n+1}}{\sqrt{k-1}}\ ,\ \frac{(-1)^n}{\sqrt{k+1}}\Big)\Big\} \end{array} \tag{5}$$

Some additional regularities appear for eigenvectors which correspond to eigenvalues other than ± 1. Example 1. The sum of all entries for each eigenvector is equal to zero. The implication is that all other eigenvectors are orthogonal to a vector with 1 in all its entries. 2. The entries in the numerator of the eigenvector corresponding to the eigenvalue zero are essentially the numbers appearing in a Pascal triangle, but with alternating signs, separated by zeros.[4] 3. The eigenvalues of the bi-graded Markovian matrix M_{2n+1} are:[5]

$$\lambda_{2n+1,\ \pm i} = \pm\sqrt{1 - \frac{i\,(i+1)}{n\,(n+1)}} \qquad \forall\ 0\ \leq\ i\ \leq\ n \tag{6}$$

The eigenvectors, being those of a (nonlocal) Dirac operator, are all spinors. One should thus not be surprised to observe the eigenvalues all appearing in pairs which only differ by their sign. This means that for every spinor, there also appears its anti-spinor (which essentially has the opposite momentum). In the case of eigenvectors which correspond to the zero eigenvalue, the spinor and its 'anti-spinor' are the same.

[2]See Appendix B.

[3]See Appendix A.

[4]See Appendix A.

[5]These eigenvalues are identical with $\pm\sqrt{\frac{\langle j,m\,|\,J_- J_+\,|j,m\,\rangle}{\langle j,m\,|\,J^2\,|j,m\,\rangle}}$ in the case of angular momentum in three dimensions, where $j \in \mathbb{N}, \quad 0 \leq m \leq j$. This should be interpreted as a geometrical average over clockwise (i.e. holomorphic) and anti-clockwise (i.e. anti-holomorphic) circular motion.

2 Some possible physical applications of bi-graded Markovian matrices

A Markovian matrix is essentially a matrix which represents a probabilistic distribution for transitions per unit "step" (possibly, a time interval), given the initial state. Each row of a Markovian matrix represents some initial state, while the columns represent the final states. The (i, j) matrix element in a Markovian matrix reproduces the probability for the system, initially in the i-state, to transfer to the j-state in one time interval. The sum of all the probabilities in each row is equal to 1. Finding the transition probabilities after n time steps is achieved by exponentiating the Markovian matrix to the n-th power. In some cases, the long-term behavior (i.e. M^n as $n \to \infty$) becomes stationary.
In the present case, starting from noncommutative geometry, we found bi-graded Markovian matrices. As one can see, in both odd and even cases, the system oscillates between two phases - which is why we describe the matrices as bi-graded. However, there is some difference between the odd and even cases. The even case oscillates between two states, both of which are Markovian, whereas the odd case oscillates asymptotically only[6], and in such a way that only one of the resulting states is Markovian.

Re-examining again the bi-graded matrices M, one sees that they can be decomposed into the sum of two matrices M_+ and M_-, where M_+ includes only the upper off-diagonal part and M_- is its transpose. The action of $M_{\{+/-\}}$ on a vector will be similar to that of raising/lowering operators. It turns out that, in the even case, the $M_{\{+/-\}}$ operators obey an anti-commutation algebra (i.e. $M_+M_- + M_-M_+ = \mathbb{I}$). However, in the odd case, the $M_{\{+/-\}}$ obey the commutation relations of a quantum algebra: $M_+M_- - QM_-M_+ = \alpha\mathbb{I}$, where Q is a rational and diagonal matrix and α is a rational number, or a Yang-Baxter like algebra $RM_+M_- = M_-M_+R$.[7]

In the following section we suggest an application in which such matrices are interpreted as describing the Markovian time evolution of the quantum states of a system.

As an example for the even case one considers a one dimensional lattice with $2n$ lattice sites. We assume that at every odd site there is a spin $J = 1/2$ particle, with $J_3 = +1/2$, and at every even site there is a $J = 1/2$ particle with $J_3 = -1/2$ (all the particles are identified). This state is represented by $\{1\ ,\ -1\ \ldots\ 1\ ,\ -1\}$.
Inspecting M_{2n} and treating it as a Markovian transition matrix which acts iteratively on states corresponding to the eigenvalue -1, we realize that the lattice is essentially split into pairs of nearest sites, with the two spin half particles (with opposite orientations) exchanging places. An alternative interpretation would consist in regarding the system as a whole as a description of a stationary spin wave.

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[6]Except in the $n = 3$ case, where the oscillations are always between two states, of which only one is Markovian.

[7]See Appendix D.

Unlike the even case, where M_{2n} itself is Markovian, in the odd case, M_{2n+1} is not a Markovian matrix; instead, it is the square root of a Markovian matrix (which is why we have chosen to denote them as bi-graded Markovian). Thus, the entries in M_{2n+1} do not represent transition probabilities. However, in M_{2n+1}^2 the entries do represent such transition probabilities. Thus, in analogy to quantum mechanics, one can identify the $i-$ row in the bi-graded Markovian matrix as a *ket* - i.e. $|\,\psi_i\,\rangle$ and the $j-$ column as a *bra* - i.e. a $\langle\psi_j\,|$. When the bi-graded Markovian matrix is squared, the $(i,j)-$ entry is $\langle\psi_j\,|\,\psi_i\,\rangle$, fitting a probabilistic interpretation. The bi-graded Markovian matrix as a whole can thus be thought of as a mixture of quantum states. The main point is that, unlike conventional quantum mechanics, where the time evolution of a quantum state is determined by the Hamiltonian, in this interpretation *the bi-graded Markovian matrices represent both the initial state and its Markovian time evolution.*
Heuristically, comparing the role of the Hamiltonian in quantum mechanics with that of a Markovian matrix in a stochastic process, we are led to the identification of the square root of a Markovian matrix (i.e. the bi-graded Markovian matrix) with a (non-local) Dirac operator. It is instructive to observe the destructive interference and the probability flows in the Markovian time evolution, in the case of bi-graded Markovian matrices.[8] It turns out that in the asymptotic regime, two states coexist - i.e. the one which associates zero probabilities to even sites, and the other, which associates zero probabilities to odd sites.

Another possible approach would consist in treating the bi-graded matrices as operators determining the time evolution of a system, as represented by a state-vector.
Due to the fact that the eigenvectors of the bi-graded matrix span a $k-$ dimensional vector space (the $k-$ dimensional Hilbert space of the spinors) the state-vector can be represented in that base. When the bi-graded matrix is exponentiated to some power m, its eigenvalues are thereby exponentiated to the same power m. However, all the eigenvalues satisfy $-1 \leq \lambda_i \leq 1$, which leads to the following asymptotic behavior:

$$\begin{array}{ll} \lambda = 1 & \Longrightarrow \lambda^\infty = 1 \\ \lambda = -1 & \Longrightarrow \lambda^\infty = \pm 1 \\ \lambda \neq \pm 1 & \Longrightarrow \lambda^\infty = 0 \end{array} \tag{7}$$

Thus, the time evolution of the state, as dictated by the bi-graded matrix, will be projected, at the very end, on two eigenvectors corresponding to the ± 1 eigenvalues. The projection on the other eigenvectors will meanwhile be gradually weakened, due to their eigenvalues approach to zero. The asymptotic state will thus consist in a superposition of two stationary states corresponding to the eigenvectors with eigenvalues $\lambda = \pm 1$). When the initial state of the system precisely coincides with the eigenvector which corresponds to $\lambda = 1$ then this state will live forever (like a soliton). If the initial state of the system was exactly the eigenvector corresponding to $\lambda = -1$, this state will be a stationary wave.

[8] See appendix C.

Only states which correspond to the ± 1 eigenvalue survive asymptotically (in the even case, this is the situation from the beginning). During the time evolution the -1 eigenvalue alternates, the time average for this eigenvalue thus tending to zero. In a measurement process with time averaging, the state corresponding to the -1 eigenvalue will thus not be observed. Note, however, that we identified the eigenvectors with spinors, which would imply that the spinor and its anti-spinor have different effective eigenvalues under time averaging. In other words, if the result of the measuring apparatus is proportional to: $\langle \psi \rangle = \frac{1}{N} \sum_{n=0}^{N} D^n \psi(0)$ we would obtain the following behavior:

$$\langle \psi \rangle_N = \begin{cases} \psi(0) & \lambda = 1 \\ \\ \pm \frac{1}{N} \psi(0) \text{ or } 0 & \lambda = -1 \end{cases}$$

This is evocative of neutrino phenomenology, in which only the left-chiral brand is observed, with the right-chiral species perhaps entirely absent. One wonders whether such considerations can be applied to quantum field theory, assuming that the spacetime substratum becomes lattice-like at Planck distances. We would then have an explanation of the absence of right-handed neutrinos "from first principles".

Another approach would consist in assuming that measurements are only sensitive to Markovian matrices (which are probabilistic) i.e. only to the even time steps. In this interpretation all eigenvalues are positive (with double degeneracy), and one can thus not distinguish between spinors and anti-spinors. This can be the case, for instance, if the relevant expectation value is defined as $\langle \psi(t_N) \,|\, \psi(t_N) \rangle = \langle \psi(0) \,|\, (D^+)^N D^N \,|\, \psi(0) \rangle$ (where $D^+ = D$).

Final remarks Although we have used the term "bi-graded Markovian" for the initial matrices only, i.e. before exponentiation, it should be clear that every odd power of these matrices shares the same property of becoming Markovian when squared.

We suggest that bi-graded Markovian matrices might be applied to a variety of new quantum and stochastic phenomena. It is thus of much importance to know more about the mathematics which govern these matrices.

As a final remark, we conjecture that it might be possible to construct $q-$graded Markovian matrices as well. It would then follow that the entries be complex numbers, which essentially represent complex probabilities. They might thus be related to noncommutative probability theory [3] and to $q-$statistics[9] [4], as well as to quantum-groups [5].

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[9]By $q-$statistics we denote particles obeying $q-$commutation relations, defined as: $[a\,,\,a^+]_q = aa^+ - qa^+a = \hbar$ where $-1 \leq q \leq 1$

partially written, and the IHES Director, for the Institute's warm hospitality.

Appendix

Appendix A. Some of the first few bi-graded Markovian matrices in the odd case, and their eigenvalues with the corresponding eigenvectors:

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{array}{l} \left\{ 1 \leftrightarrow \left(\frac{1}{2} \, , \, \frac{1}{\sqrt{2}} \, , \, \frac{1}{2} \right) \right\} \\ \left\{ -1 \leftrightarrow \left(-\frac{1}{2} \, , \, \frac{1}{\sqrt{2}} \, , \, -\frac{1}{2} \right) \right\} \\ \left\{ 0 \leftrightarrow \left(-\frac{1}{\sqrt{2}} \, , \, 0 \, , \, \frac{1}{\sqrt{2}} \right) \right\} \end{array}$$

$$\begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 0 \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} \\ 0 & 0 & 0 & \frac{2}{\sqrt{6}} & 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{array}{l} \left\{ 1 \leftrightarrow \left(\frac{1}{\sqrt{6}} \, , \, \frac{1}{2} \, , \, \frac{1}{\sqrt{6}} \, , \, \frac{1}{2} \, , \, \frac{1}{\sqrt{6}} \right) \right\} \\ \left\{ -1 \leftrightarrow \left(\frac{1}{\sqrt{6}} \, , \, -\frac{1}{2} \, , \, \frac{1}{\sqrt{6}} \, , \, -\frac{1}{2} \, , \, \frac{1}{\sqrt{6}} \right) \right\} \\ \left\{ 0 \leftrightarrow \left(\frac{1}{\sqrt{6}} \, , \, 0 \, , \, -\frac{2}{\sqrt{6}} \, , \, 0 \, , \, \frac{1}{\sqrt{6}} \right) \right\} \\ \left\{ -\sqrt{\frac{2}{3}} \leftrightarrow \left(-\frac{1}{2} \, , \, \frac{1}{2} \, , \, 0 \, , \, -\frac{1}{2} \, , \, \frac{1}{2} \right) \right\} \\ \left\{ \sqrt{\frac{2}{3}} \leftrightarrow \left(-\frac{1}{2} \, , \, -\frac{1}{2} \, , \, 0 \, , \, \frac{1}{2} \, , \, \frac{1}{2} \right) \right\} \end{array}$$

$$\begin{pmatrix} 0 & \frac{3}{\sqrt{12}} & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{\sqrt{12}} & 0 & \frac{1}{\sqrt{12}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{12}} & 0 & \frac{2}{\sqrt{12}} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{12}} & 0 & \frac{2}{\sqrt{12}} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\sqrt{12}} & 0 & \frac{1}{\sqrt{12}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{12}} & 0 & \frac{3}{\sqrt{12}} \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{\sqrt{12}} & 0 \end{pmatrix} \qquad \Longrightarrow$$

$$\begin{array}{l} \left\{ 1 \leftrightarrow \left(\frac{1}{\sqrt{8}} \, , \, \frac{1}{\sqrt{6}} \, , \, \frac{1}{\sqrt{8}} \, , \, \frac{1}{\sqrt{6}} \, , \, \frac{1}{\sqrt{8}} \, , \, \frac{1}{\sqrt{6}} \, , \, \frac{1}{\sqrt{8}} \right) \right\} \\ \left\{ -1 \leftrightarrow \left(-\frac{1}{\sqrt{8}} \, , \, \frac{1}{\sqrt{6}} \, , \, -\frac{1}{\sqrt{8}} \, , \, \frac{1}{\sqrt{6}} \, , \, -\frac{1}{\sqrt{8}} \, , \, \frac{1}{\sqrt{6}} \, , \, -\frac{1}{\sqrt{8}} \right) \right\} \\ \left\{ 0 \leftrightarrow \left(-\frac{1}{\sqrt{8}} \, , \, 0 \, , \, \frac{3}{\sqrt{8}} \, , \, 0 \, , \, -\frac{3}{\sqrt{8}} \, , \, 0 \, , \, \frac{1}{\sqrt{8}} \right) \right\} \\ \left\{ \sqrt{\frac{5}{6}} \leftrightarrow \left(-\frac{3}{\sqrt{40}} \, , \, -\frac{1}{2} \, , \, -\frac{1}{\sqrt{40}} \, , \, 0 \, , \, \frac{1}{\sqrt{40}} \, , \, \frac{1}{2} \, , \, \frac{3}{\sqrt{40}} \right) \right\} \\ \left\{ -\sqrt{\frac{5}{6}} \leftrightarrow \left(-\frac{3}{\sqrt{40}} \, , \, \frac{1}{2} \, , \, -\frac{1}{\sqrt{40}} \, , \, 0 \, , \, \frac{1}{\sqrt{40}} \, , \, -\frac{1}{2} \, , \, \frac{3}{\sqrt{40}} \right) \right\} \\ \left\{ \frac{1}{\sqrt{2}} \leftrightarrow \left(-\frac{1}{\sqrt{8}} \, , \, -\frac{1}{\sqrt{12}} \, , \, \frac{1}{\sqrt{8}} \, , \, \frac{1}{\sqrt{3}} \, , \, \frac{1}{\sqrt{8}} \, , \, -\frac{1}{\sqrt{12}} \, , \, -\frac{1}{\sqrt{8}} \right) \right\} \\ \left\{ -\frac{1}{\sqrt{2}} \leftrightarrow \left(-\frac{1}{\sqrt{8}} \, , \, \frac{1}{\sqrt{12}} \, , \, \frac{1}{\sqrt{8}} \, , \, -\frac{1}{\sqrt{3}} \, , \, \frac{1}{\sqrt{8}} \, , \, \frac{1}{\sqrt{12}} \, , \, -\frac{1}{\sqrt{8}} \right) \right\} \end{array}$$

$$\begin{pmatrix} 0 & \frac{4}{\sqrt{20}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{\sqrt{20}} & 0 & \frac{1}{\sqrt{20}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{20}} & 0 & \frac{3}{\sqrt{20}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{\sqrt{20}} & 0 & \frac{2}{\sqrt{20}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\sqrt{20}} & 0 & \frac{2}{\sqrt{20}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\sqrt{20}} & 0 & \frac{3}{\sqrt{20}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{\sqrt{20}} & 0 & \frac{1}{\sqrt{20}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{20}} & 0 & \frac{4}{\sqrt{20}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{\sqrt{20}} & 0 \end{pmatrix} \qquad \Longrightarrow$$

$$\left\{ \quad 0 \leftrightarrow \left(\frac{1}{\sqrt{70}} \; , \; 0 \; , \; -\frac{4}{\sqrt{70}} \; , \; 0 \; , \; \frac{6}{\sqrt{70}} \; , \; 0 \; , \; -\frac{4}{\sqrt{70}} \; , \; 0 \; , \; \frac{1}{\sqrt{70}} \right) \right\}$$

$$\left\{ -1 \leftrightarrow \left(\frac{1}{\sqrt{10}} \; , \; -\frac{1}{\sqrt{8}} \; , \; \frac{1}{\sqrt{10}} \; , \; -\frac{1}{\sqrt{8}} \; , \; \frac{1}{\sqrt{10}} \; , \; -\frac{1}{\sqrt{8}} \; , \; \frac{1}{\sqrt{10}} \; , \; -\frac{1}{\sqrt{8}} \; , \; \frac{1}{\sqrt{10}} \right) \right\}$$

$$\left\{ \quad 1 \leftrightarrow \left(\frac{1}{\sqrt{10}} \; , \; \frac{1}{\sqrt{8}} \; , \; \frac{1}{\sqrt{10}} \; , \; \frac{1}{\sqrt{8}} \; , \; \frac{1}{\sqrt{10}} \; , \; \frac{1}{\sqrt{8}} \; , \; \frac{1}{\sqrt{10}} \; , \; \frac{1}{\sqrt{8}} \; , \; \frac{1}{\sqrt{10}} \right) \right\}$$

$$\left\{ \quad \sqrt{\frac{7}{10}} \leftrightarrow \left(\frac{1}{\sqrt{7}} \; , \; \frac{1}{\sqrt{80}} \; , \; -\frac{1}{\sqrt{28}} \; , \; -\frac{1}{\sqrt{80}} \; , \; -\frac{1}{\sqrt{7}} \; , \; -\frac{1}{\sqrt{80}} \; , \; -\frac{1}{\sqrt{28}} \; , \; \frac{1}{\sqrt{80}} \; , \; \frac{1}{\sqrt{7}} \right) \right\}$$

$$\left\{ -\sqrt{\frac{7}{10}} \leftrightarrow \left(\frac{1}{\sqrt{7}} \; , \; -\frac{1}{\sqrt{80}} \; , \; -\frac{1}{\sqrt{28}} \; , \; \frac{1}{\sqrt{80}} \; , \; -\frac{1}{\sqrt{7}} \; , \; \frac{1}{\sqrt{80}} \; , \; -\frac{1}{\sqrt{28}} \; , \; -\frac{1}{\sqrt{80}} \; , \; \frac{1}{\sqrt{7}} \right) \right\}$$

$$\left\{ \quad \sqrt{\frac{2}{5}} \leftrightarrow \left(-\frac{1}{\sqrt{20}} \; , \; -\frac{1}{\sqrt{40}} \; , \; \frac{2}{\sqrt{20}} \; , \; \frac{3}{\sqrt{40}} \; , \; 0 \; , \; -\frac{3}{\sqrt{40}} \; , \; -\frac{2}{\sqrt{20}} \; , \; \frac{1}{\sqrt{40}} \; , \; \frac{1}{\sqrt{20}} \right) \right\}$$

$$\left\{ -\sqrt{\frac{2}{5}} \leftrightarrow \left(\frac{1}{\sqrt{20}} \; , \; -\frac{1}{\sqrt{40}} \; , \; -\frac{2}{\sqrt{20}} \; , \; \frac{3}{\sqrt{40}} \; , \; 0 \; , \; -\frac{3}{\sqrt{40}} \; , \; \frac{2}{\sqrt{20}} \; , \; \frac{1}{\sqrt{40}} \; , \; -\frac{1}{\sqrt{20}} \right) \right\}$$

$$\left\{ \quad \frac{3}{\sqrt{10}} \leftrightarrow \left(\frac{1}{\sqrt{5}} \; , \; \frac{3}{\sqrt{40}} \; , \; \frac{1}{\sqrt{20}} \; , \; \frac{1}{\sqrt{40}} \; , \; 0 \; , \; -\frac{1}{\sqrt{40}} \; , \; -\frac{1}{\sqrt{20}} \; , \; -\frac{3}{\sqrt{40}} \; , \; -\frac{1}{\sqrt{5}} \right) \right\}$$

$$\left\{ -\frac{3}{\sqrt{10}} \leftrightarrow \left(\frac{1}{\sqrt{5}} \; , \; -\frac{3}{\sqrt{40}} \; , \; \frac{1}{\sqrt{20}} \; , \; -\frac{1}{\sqrt{40}} \; , \; 0 \; , \; \frac{1}{\sqrt{40}} \; , \; -\frac{1}{\sqrt{20}} \; , \; \frac{3}{\sqrt{40}} \; , \; -\frac{1}{\sqrt{5}} \right) \right\}$$

Appendix B. Some of the first few bi-graded Markovian matrices in the even case, and their eigenvalues with the corresponding eigenvectors:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow \{1 \leftrightarrow (1\ ,\ 1)\}\,,\{-1 \leftrightarrow (-1\ ,\ 1)\}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{array}{ll} \{1 \leftrightarrow (1\ ,\ 1\ ,\ 0\ ,\ 0)\} & \{-1 \leftrightarrow (-1\ ,\ 1\ ,\ 0\ ,\ 0)\} \\ \{1 \leftrightarrow (0\ ,\ 0\ ,\ 1\ ,\ 1)\} & \{-1 \leftrightarrow (0\ ,\ 0\ ,\ -1\ ,\ 1)\} \end{array}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \Longrightarrow$$

$$\begin{array}{ll} \{1 \leftrightarrow (1\ ,\ 1\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0)\} & \{-1 \leftrightarrow (-1\ ,\ 1\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0)\} \\ \{1 \leftrightarrow (0\ ,\ 0\ ,\ 1\ ,\ 1\ ,\ 0\ ,\ 0)\} & \{-1 \leftrightarrow (0\ ,\ 0\ ,\ -1\ ,\ 1\ ,\ 0\ ,\ 0)\} \\ \{1 \leftrightarrow (0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 1\ ,\ 1)\} & \{-1 \leftrightarrow (0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ -1\ ,\ 1)\} \end{array}$$

Appendix C. An example for a bi-graded matrix evolution in the odd case:

$$A = \begin{pmatrix} 0 & \frac{4}{\sqrt{20}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{\sqrt{20}} & 0 & \frac{1}{\sqrt{20}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{20}} & 0 & \frac{3}{\sqrt{20}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{\sqrt{20}} & 0 & \frac{2}{\sqrt{20}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\sqrt{20}} & 0 & \frac{2}{\sqrt{20}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\sqrt{20}} & 0 & \frac{3}{\sqrt{20}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{\sqrt{20}} & 0 & \frac{1}{\sqrt{20}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{20}} & 0 & \frac{4}{\sqrt{20}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{\sqrt{20}} & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} .8 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .85 & 0 & .15 & 0 & 0 & 0 & 0 & 0 \\ .2 & 0 & .5 & 0 & .3 & 0 & 0 & 0 & 0 \\ 0 & .15 & 0 & .65 & 0 & .2 & 0 & 0 & 0 \\ 0 & 0 & .3 & 0 & .4 & 0 & .3 & 0 & 0 \\ 0 & 0 & 0 & .2 & 0 & .65 & 0 & .15 & 0 \\ 0 & 0 & 0 & 0 & .3 & 0 & .5 & 0 & .2 \\ 0 & 0 & 0 & 0 & 0 & .15 & 0 & .85 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .2 & 0 & .8 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & .76026 & 0 & .13416 & 0 & 0 & 0 & 0 & 0 \\ .76026 & 0 & .29069 & 0 & .06708 & 0 & 0 & 0 & 0 \\ 0 & .29069 & 0 & .46957 & 0 & .13416 & 0 & 0 & 0 \\ .13416 & 0 & .46957 & 0 & .38013 & 0 & .13416 & 0 & 0 \\ 0 & .06708 & 0 & .38013 & 0 & .38013 & 0 & .06708 & 0 \\ 0 & 0 & .13416 & 0 & .38013 & 0 & .46957 & 0 & .13416 \\ 0 & 0 & 0 & .13416 & 0 & .46957 & 0 & .29069 & 0 \\ 0 & 0 & 0 & 0 & .06708 & 0 & .29069 & 0 & .76026 \\ 0 & 0 & 0 & 0 & 0 & .13416 & 0 & .76026 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} .68 & 0 & .26 & 0 & .06 & 0 & 0 & 0 & 0 \\ 0 & .745 & 0 & .225 & 0 & .03 & 0 & 0 & 0 \\ .26 & 0 & .38 & 0 & .27 & 0 & .09 & 0 & 0 \\ 0 & .225 & 0 & .485 & 0 & .26 & 0 & .03 & 0 \\ .06 & 0 & .27 & 0 & .34 & 0 & .27 & 0 & .06 \\ 0 & .03 & 0 & .26 & 0 & .485 & 0 & .225 & 0 \\ 0 & 0 & .09 & 0 & .27 & 0 & .38 & 0 & .26 \\ 0 & 0 & 0 & .03 & 0 & .225 & 0 & .745 & 0 \\ 0 & 0 & 0 & 0 & .06 & 0 & .26 & 0 & .68 \end{pmatrix}$$

$$A^7 = \begin{pmatrix} 0 & .59658 & 0 & .23613 & 0 & .05769 & 0 & .00405 & 0 \\ .59658 & 0 & .32624 & 0 & .14691 & 0 & .04427 & 0 & .00405 \\ 0 & .32624 & 0 & .32803 & 0 & .19588 & 0 & .04427 & 0 \\ .23613 & 0 & .32803 & 0 & .3003 & 0 & .19588 & 0 & .05769 \\ 0 & .14691 & 0 & .3003 & 0 & .3003 & 0 & .14691 & 0 \\ .05769 & 0 & .19588 & 0 & .3003 & 0 & .32803 & 0 & .23613 \\ 0 & .04427 & 0 & .19588 & 0 & .32803 & 0 & .32624 & 0 \\ .00405 & 0 & .04427 & 0 & .14691 & 0 & .32624 & 0 & .59658 \\ 0 & .00405 & 0 & .05769 & 0 & .23613 & 0 & .59658 & 0 \end{pmatrix}$$

$$A^8 = \begin{pmatrix} .5336 & 0 & .2918 & 0 & .1314 & 0 & .0396 & 0 & .0036 \\ 0 & .60655 & 0 & .28455 & 0 & .0954 & 0 & .0135 & 0 \\ .2918 & 0 & .293 & 0 & .2343 & 0 & .1413 & 0 & .0396 \\ 0 & .28455 & 0 & .35435 & 0 & .2657 & 0 & .0954 & 0 \\ .1314 & 0 & .2343 & 0 & .2686 & 0 & .2343 & 0 & .1314 \\ 0 & .0954 & 0 & .2657 & 0 & .35435 & 0 & .28455 & 0 \\ .0396 & 0 & .1413 & 0 & .2343 & 0 & .293 & 0 & .2918 \\ 0 & .0135 & 0 & .0954 & 0 & .28455 & 0 & .60655 & 0 \\ .0036 & 0 & .0396 & 0 & .1314 & 0 & .2918 & 0 & .5336 \end{pmatrix}$$

$$A^{15} = \begin{pmatrix} 0 & .43461 & 0 & .26914 & 0 & .14124 & 0 & .04944 & 0 \\ .43461 & 0 & .31051 & 0 & .20519 & 0 & .11829 & 0 & .04944 \\ 0 & .31051 & 0 & .26534 & 0 & .20029 & 0 & .11829 & 0 \\ .26914 & 0 & .26534 & 0 & .24202 & 0 & .20029 & 0 & .14124 \\ 0 & .20519 & 0 & .24202 & 0 & .24202 & 0 & .20519 & 0 \\ .14124 & 0 & .20029 & 0 & .24202 & 0 & .26534 & 0 & .26914 \\ 0 & .11829 & 0 & .20029 & 0 & .26534 & 0 & .31051 & 0 \\ .04944 & 0 & .11829 & 0 & .20519 & 0 & .31051 & 0 & .43461 \\ 0 & .04944 & 0 & .14124 & 0 & .26914 & 0 & .43461 & 0 \end{pmatrix}$$

$$A^{16} = \begin{pmatrix} .38872 & 0 & .27773 & 0 & .18353 & 0 & .1058 & 0 & .04422 \\ 0 & .45816 & 0 & .30006 & 0 & .17112 & 0 & .07067 & 0 \\ .27773 & 0 & .24743 & 0 & .20824 & 0 & .16081 & 0 & .1058 \\ 0 & .30006 & 0 & .28623 & 0 & .24259 & 0 & .17112 & 0 \\ .18353 & 0 & .20824 & 0 & .21647 & 0 & .20824 & 0 & .18353 \\ 0 & .17112 & 0 & .24259 & 0 & .28623 & 0 & .30006 & 0 \\ .1058 & 0 & .16081 & 0 & .20824 & 0 & .24743 & 0 & .27773 \\ 0 & .07067 & 0 & .17112 & 0 & .30006 & 0 & .45816 & 0 \\ .04422 & 0 & .1058 & 0 & .18353 & 0 & .27773 & 0 & .38872 \end{pmatrix}$$

$$A^{31} = \begin{pmatrix} 0 & .30754 & 0 & .25017 & 0 & .19492 & 0 & .1418 & 0 \\ .30754 & 0 & .26451 & 0 & .22255 & 0 & .18164 & 0 & .1418 \\ 0 & .26451 & 0 & .23795 & 0 & .21033 & 0 & .18164 & 0 \\ .25017 & 0 & .23795 & 0 & .22467 & 0 & .21033 & 0 & .19492 \\ 0 & .22255 & 0 & .22467 & 0 & .22467 & 0 & .22255 & 0 \\ .19492 & 0 & .21033 & 0 & .22467 & 0 & .23795 & 0 & .25017 \\ 0 & .18164 & 0 & .21033 & 0 & .23795 & 0 & .26451 & 0 \\ .1418 & 0 & .18164 & 0 & .22255 & 0 & .26451 & 0 & .30754 \\ 0 & .1418 & 0 & .19492 & 0 & .25017 & 0 & .30754 & 0 \end{pmatrix}$$

$$A^{32} = \begin{pmatrix} .27507 & 0 & .23659 & 0 & .19905 & 0 & .16246 & 0 & .12683 \\ 0 & .33422 & 0 & .27696 & 0 & .22137 & 0 & .16744 & 0 \\ .23659 & 0 & .21877 & 0 & .20047 & 0 & .18171 & 0 & .16246 \\ 0 & .27696 & 0 & .2601 & 0 & .24157 & 0 & .22137 & 0 \\ .19905 & 0 & .20047 & 0 & .20095 & 0 & .20047 & 0 & .19905 \\ 0 & .22137 & 0 & .24157 & 0 & .2601 & 0 & .27696 & 0 \\ .16246 & 0 & .18171 & 0 & .20047 & 0 & .21877 & 0 & .23659 \\ 0 & .16744 & 0 & .22137 & 0 & .27696 & 0 & .33422 & 0 \\ .12683 & 0 & .16246 & 0 & .19905 & 0 & .23659 & 0 & .27507 \end{pmatrix}$$

$$A^{63} = \begin{pmatrix} 0 & .23897 & 0 & .22872 & 0 & .21848 & 0 & .20825 & 0 \\ .23897 & 0 & .23128 & 0 & .2236 & 0 & .21593 & 0 & .20825 \\ 0 & .23128 & 0 & .22617 & 0 & .22105 & 0 & .21593 & 0 \\ .22872 & 0 & .22617 & 0 & .22361 & 0 & .22105 & 0 & .21848 \\ 0 & .2236 & 0 & .22361 & 0 & .22361 & 0 & .2236 & 0 \\ .21848 & 0 & .22105 & 0 & .22361 & 0 & .22617 & 0 & .22872 \\ 0 & .21593 & 0 & .22105 & 0 & .22617 & 0 & .23128 & 0 \\ .20825 & 0 & .21593 & 0 & .2236 & 0 & .23128 & 0 & .23897 \\ 0 & .20825 & 0 & .21848 & 0 & .22872 & 0 & .23897 & 0 \end{pmatrix}$$

$$A^{64} = \begin{pmatrix} .21374 & 0 & .20687 & 0 & .2 & 0 & .19313 & 0 & .18627 \\ 0 & .26545 & 0 & .25515 & 0 & .24485 & 0 & .23455 & 0 \\ .20687 & 0 & .20343 & 0 & .2 & 0 & .19657 & 0 & .19313 \\ 0 & .25515 & 0 & .25172 & 0 & .24829 & 0 & .24485 & 0 \\ .2 & 0 & .2 & 0 & .2 & 0 & .2 & 0 & .2 \\ 0 & .24485 & 0 & .24829 & 0 & .25172 & 0 & .25515 & 0 \\ .19313 & 0 & .19657 & 0 & .2 & 0 & .20343 & 0 & .20687 \\ 0 & .23455 & 0 & .24485 & 0 & .25515 & 0 & .26545 & 0 \\ .18627 & 0 & .19313 & 0 & .2 & 0 & .20687 & 0 & .21374 \end{pmatrix}$$

$$A^{127} = \begin{pmatrix} 0 & .22413 & 0 & .22378 & 0 & .22343 & 0 & .22308 & 0 \\ .22413 & 0 & .22387 & 0 & .22361 & 0 & .22334 & 0 & .22308 \\ 0 & .22387 & 0 & .22369 & 0 & .22352 & 0 & .22334 & 0 \\ .22378 & 0 & .22369 & 0 & .22361 & 0 & .22352 & 0 & .22343 \\ 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 \\ .22343 & 0 & .22352 & 0 & .22361 & 0 & .22369 & 0 & .22378 \\ 0 & .22334 & 0 & .22352 & 0 & .22369 & 0 & .22387 & 0 \\ .22308 & 0 & .22334 & 0 & .22361 & 0 & .22387 & 0 & .22413 \\ 0 & .22308 & 0 & .22343 & 0 & .22378 & 0 & .22413 & 0 \end{pmatrix}$$

$$A^{128} = \begin{pmatrix} .20047 & 0 & .20024 & 0 & .2 & 0 & .19976 & 0 & .19953 \\ 0 & .25053 & 0 & .25018 & 0 & .24982 & 0 & .24947 & 0 \\ .20024 & 0 & .20012 & 0 & .2 & 0 & .19988 & 0 & .19976 \\ 0 & .25018 & 0 & .25006 & 0 & .24994 & 0 & .24982 & 0 \\ .2 & 0 & .2 & 0 & .2 & 0 & .2 & 0 & .2 \\ 0 & .24982 & 0 & .24994 & 0 & .25006 & 0 & .25018 & 0 \\ .19976 & 0 & .19988 & 0 & .2 & 0 & .20012 & 0 & .20024 \\ 0 & .24947 & 0 & .24982 & 0 & .25018 & 0 & .25053 & 0 \\ .19953 & 0 & .19976 & 0 & .2 & 0 & .20024 & 0 & .20047 \end{pmatrix}$$

$$A^{255} = \begin{pmatrix} 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 \\ .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 \\ 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 \\ .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 \\ 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 \\ .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 \\ 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 \\ .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 \\ 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 & .22361 & 0 \end{pmatrix}$$

$$A^{256} = \begin{pmatrix} .2 & 0 & .2 & 0 & .2 & 0 & .2 & 0 & .2 \\ 0 & .25 & 0 & .25 & 0 & .25 & 0 & .25 & 0 \\ .2 & 0 & .2 & 0 & .2 & 0 & .2 & 0 & .2 \\ 0 & .25 & 0 & .25 & 0 & .25 & 0 & .25 & 0 \\ .2 & 0 & .2 & 0 & .2 & 0 & .2 & 0 & .2 \\ 0 & .25 & 0 & .25 & 0 & .25 & 0 & .25 & 0 \\ .2 & 0 & .2 & 0 & .2 & 0 & .2 & 0 & .2 \\ 0 & .25 & 0 & .25 & 0 & .25 & 0 & .25 & 0 \\ .2 & 0 & .2 & 0 & .2 & 0 & .2 & 0 & .2 \end{pmatrix}$$

Appendix D. The Algebra which governs M_+ and M_- :

$M_{+/-}$ are the raising and lowering operators, which together build the bi-graded Markovian matrices. Q & R are matrices. The entries in the R matrix which are denoted by $a_{i,j}$ are free. The Q matrix is diagonal and rational.

$$M_{2+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad M_{2-} = (M_{2+})^{tr} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$M_{2+}M_{2-} - M_{2-}M_{2+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M_{2+}M_{2-} + M_{2-}M_{2+} = \mathbb{I}_2$$

$$M_{4+} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{4-} = (M_{4+})^{tr} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M_{4+}M_{4-} - M_{4-}M_{4+} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$M_{4+}M_{4-} + M_{4-}M_{4+} = \mathbb{I}_4$$

$$M_{6+} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{6-} = (M_{6+})^{tr}$$

$$M_{6+}M_{6-} - M_{6-}M_{6+} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$M_{6+}M_{6-} + M_{6-}M_{6+} = \mathbb{I}_6$$

$$M_{3+} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix} \qquad M_{3-} = (M_{3+})^{tr}$$

$$R_3 = \begin{pmatrix} 0 & 0 & a_{1,3} \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & 0 \end{pmatrix} \quad \Longrightarrow \quad R_3 M_{3+} M_{3-} = M_{3-} M_{3+} R_3$$

$$M_{3+}M_{3-} - M_{3-}M_{3+} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

$$M_{3+}M_{3-} + M_{3-}M_{3+} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$Q_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad M_{3+}M_{3-} + Q_3 M_{3-} M_{3+} = \tfrac{1}{2}\mathbb{I}_3$$

$$M_{5+} = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{5-} = (M_{5+})^{tr}$$

$$R_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & a_{1,5} \\ a_{2,1} & 0 & 0 & a_{2,4} & 0 \\ 0 & a_{3,2} & a_{3,3} & 0 & 0 \\ 0 & a_{4,2} & a_{4,3} & 0 & 0 \\ a_{5,1} & 0 & 0 & a_{5,4} & 0 \end{pmatrix} \quad \Longrightarrow \quad R_5 M_{5+} M_{5-} = M_{5-} M_{5+} R_5$$

$$M_{5+}M_{5-} - M_{5-}M_{5+} = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

$$M_{5+}M_{5-} + M_{5-}M_{5+} = \begin{pmatrix} \frac{4}{9} & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{12} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

$$Q_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad M_{5+}M_{5-} + Q_5 M_{5-} M_{5+} = \tfrac{2}{3}\mathbb{I}_5$$

$$M_{7+} = \begin{pmatrix} 0 & \frac{3}{\sqrt{12}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{12}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\sqrt{12}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\sqrt{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{\sqrt{12}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{7-} = (M_{7+})^{tr}$$

$$R_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a_{1,7} \\ a_{2,1} & 0 & 0 & 0 & 0 & a_{2,6} & 0 \\ 0 & a_{3,2} & 0 & 0 & a_{3,5} & 0 & 0 \\ 0 & 0 & a_{4,3} & a_{4,4} & 0 & 0 & 0 \\ 0 & 0 & a_{5,3} & a_{5,4} & 0 & 0 & 0 \\ 0 & a_{6,2} & 0 & 0 & a_{6,5} & 0 & 0 \\ a_{7,1} & 0 & 0 & 0 & 0 & a_{7,6} & 0 \end{pmatrix} \quad \Longrightarrow$$

$$\Longrightarrow \quad R_7 M_{7+} M_{7-} = M_{7-} M_{7+} R_7$$

$$M_{7+}M_{7-} - M_{7-}M_{7+} = \begin{pmatrix} \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} \end{pmatrix}$$

$$M_{7+}M_{7-} + M_{7-}M_{7+} = \begin{pmatrix} \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}$$

$$Q_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad M_{7+}M_{7-} + Q_7 M_{7-}M_{7+} = \frac{3}{4}\mathbb{I}_7$$

$$M_{9+} = \begin{pmatrix} 0 & \frac{4}{\sqrt{20}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{20}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{\sqrt{20}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\sqrt{20}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{20}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{\sqrt{20}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{20}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{\sqrt{20}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{9-} = (M_{9+})^{tr}$$

$$R_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1,9} \\ a_{2,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{2,8} & 0 \\ 0 & a_{3,2} & 0 & 0 & 0 & 0 & a_{3,7} & 0 & 0 \\ 0 & 0 & a_{4,3} & 0 & 0 & a_{4,6} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{5,4} & a_{5,5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{6,4} & a_{6,5} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{7,3} & 0 & 0 & a_{7,6} & 0 & 0 & 0 \\ 0 & a_{8,2} & 0 & 0 & 0 & 0 & a_{8,7} & 0 & 0 \\ a_{9,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{9,8} & 0 \end{pmatrix} \implies$$

$$\implies \quad R_9 M_{9+} M_{9-} = M_{9-} M_{9+} R_9$$

$$M_{9+}M_{9-} - M_{9-}M_{9+} = \begin{pmatrix} \frac{4}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{4}{5} \end{pmatrix}$$

$$M_{9+}M_{9-} + M_{9-}M_{9+} = \begin{pmatrix} \frac{4}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{17}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{13}{20} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{13}{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{17}{20} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{5} \end{pmatrix}$$

$$Q_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{15}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{7}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \implies \quad M_{9+}M_{9-} + Q_9 M_{9-}M_{9+} = \frac{4}{5}\mathbb{I}_9$$

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